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An internal solitary wave of large amplitude*

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Abstract: A theory of a finite-amplitude internal solitary wave in a two-fluid system is presented. Both the profile and the dispersion relation of the wave are different from those of the KdV or the Benjamin-Ono theory. Particular attention is paid to the fact that the present theory is valid for amplitude larger than the KdV soliton and the former is a generalization of the latter in a two-fluid system.

1. Introduction

Finite amplitude effects on internal waves have been studied by numerous investigators. Among other theoretical works, BENNY (1966) showed that the shallow water motions in two layers are governed by the Korteweg-de Vries equation. This theory was extended to include higher order non-linear effects by KOOP and BUTLER (1981). In addition to the shallow water theories, non-linear internal wave motion in fluids of infinite extent was analyzed by BENJAMIN (1967) and ONO (1975). These shallow-water and deep-water problems have solitary wave solutions (The formulae are given in the Appendix). JOEPH (1977) and KUBOTA et al. (1978) presented the finite-depth theory which connects these two problems.

All of the above theories are based on the expansion parameter $\delta = A/h_1$ where A is the wave amplitude and h_1 is the smaller of the two fluid depths, and therefore their results are restricted to the case when δ is small compared with unity. However, some numerical calculations (PULLIN and GRIMSHAW, 2983, FUNAKOSHI and OIKAWA, 1984) as well as oceanographic observations (e.g. SANDSTROM and ELLIOTT, 1984) suggest the existence of large-amplitude ($\delta > 1$) solitary waves. The main object of the present paper is to derive an analytical solution of a steady large-amplitude solitary wave in a two-fluid system.

2. Governing equations

Consider steady irrotational incompressible two-dimensional flow in a two-layer system bounded above and below by rigid horizontal walls. Both lower and upper fluids are homogeneous with densities ρ_1 and ρ_2 and the depths h_1 and h_2 (which is assumed to be greater than h_1), respectively (Fig. 1). The coordinate origin is at a point on the undisturbed boundary between the two layers with x in the direction of flow. The gravitational acceleration g acts in the z-direction. The wave train set-up is stationary in the (x, z) plane, with fluid velocity components (u_1, v_1) in the lower layer and (u_2, v_2) in the upper layer.

Now, in any steady two-dimensional flow, the volume flow rate per unit span in the absence of friction has the same value at every cross-section of the flow. That is, if we define

$$Q_1 = \int_{-h_1}^{\eta} u_1 dz\,, \qquad (1)$$

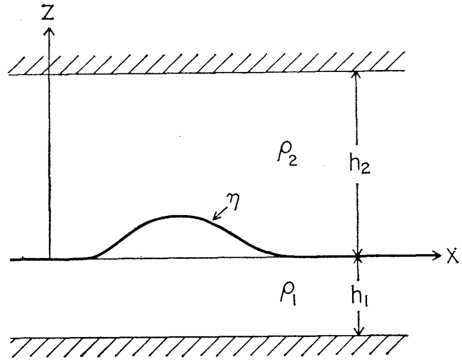

Fig. 1. The two-layer fluid system under consideration.

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$$Q_2=\int_{\eta}^{h_2} u_2 dz\,, \tag{2}$$

then each of Q_1 and Q_2 should be constant. Also, Bernouilli's equations for the two layers with datum taken at $z=0$ are:

$$R_1=p_1+\rho_1 gz+\frac{\rho_1}{2}(u_1^2+v_1^2)\,, \tag{3}$$

$$R_2=p_2+\rho_2 gz+\frac{\rho_2}{2}(u_2^2+v_2^2)\,, \tag{4}$$

where p_1 and p_2 are pressure and R_1 and R_2 are constant. In addition, the flow force (or the momentum flow rate per unit span, corrected for changes in horizontal pressure force) must be conserved. Therefore, if we write

$$S=\int_{-h_1}^{\eta}(p_1+\rho_1 u_1^2)dz+\int_{\eta}^{h_2}(p_2+\rho_2 u_2^2)dz\,, \tag{5}$$

then S is also constant.

Substituting (3) and (4) into (5) and integrating, we obtain

$$S-R_1(\eta+h_1)+R_2(\eta-h_2)+\frac{1}{2}\rho_1 g(\eta^2-h_1^2)+\frac{1}{2}\rho_2 g(h_2^2-\eta^2)=\frac{\rho_1}{2}\int_{-h_1}^{\eta}(u_1^2-v_1^2)dz+\frac{\rho_2}{2}\int_{\eta}^{h_2}(u_2^2-v_2^2)\,dz\,. \tag{6}$$

Since in each layer a complex potential w_j exists, we have

$$w_j(m)=\phi_j+i\psi_j,\ m=x+iz,\ (j=1 \text{ or } 2), \tag{7}$$

where ϕ_j and ψ_j are velocity potential and stream function, so that

$$\frac{dw_j}{dm}=\frac{\partial\phi_j}{\partial x}+i\frac{\partial\psi_j}{\partial x}=\frac{\partial\psi_j}{\partial z}-i\frac{\partial\phi_j}{\partial z}=u_j-iv_j\,, \tag{8}$$

w_j should be an analytic function which satisfies the boundary condition,

$$v_1(x,\ -h_1)=v_2(x,\ h_2)=0. \tag{9}$$

Such a function can be given by

$$\frac{dw_j}{dm}=e^{izD}\,u_j\,(x,\ h),\ (h=-h_1 \text{ or } h_2), \tag{10}$$

where D is an operator,

$$e^{izD}=1+i(z-h)\frac{d}{dx}+\frac{1}{2!}i^2(z-h)^2\frac{d^2}{dx^2}+\frac{1}{3!}i^3(z-h)_8\frac{d^3}{dx^3}+\cdots\cdots. \tag{11}$$

Equation (1) can be expressed as

$$Q_1=\int_{h_1}^{\eta}\left[1-\frac{(z+h_1)^2}{2!}\frac{d^2}{dx^2}+\frac{(z+h_1)^4}{4!}\frac{d^4}{dx^4}+\cdots\cdots\right]u_1(x,\ -h_1)\,dz=(\eta+h_1)\times\left(1-\frac{(\eta+h_1)^2}{3!}\frac{d^2}{dx^2}\right)u_1(x,\ -h_1)\,. \tag{12}$$

It should be noted that in deriving Eq. (12) the fifth and succeeding terms in Eq. (11) were neglected. This means that the following assumption was made

$$\left(\frac{A+h_1}{L}\right)^4\ll 1, \tag{13}$$

where L is the horizontal scale of the wave and A is the amplitude. Within the same approximation, Eq. (12) can be changed to:

$$u_1(x,\ -h_1)=\left(1+\frac{(\eta+h_1)^2}{3!}\frac{d^2}{dx^2}\right)\frac{Q_1}{\eta+h_1}. \tag{14}$$

In the same way, if

$$\left(\frac{h_2-A}{L}\right)^4\ll 1 \tag{15}$$

is assumed, we can obtain, approximately,

$$u_2(x,\ h_2)=\left(1+\frac{(\eta-h_2)^2}{3!}\frac{d^2}{dx^2}\right)\frac{Q_2}{\eta-h_2}. \tag{16}$$

Using Eq's (10), (14) and (16), the right-hand side of Eq. (6) can be integrated. The result is:

$$S-R_1(\eta+h_1)+R_2(\eta-h_2)+\frac{1}{2}\rho_1 g(\eta^2-h_1^2)+\frac{1}{2}\rho_2 g(h_2^2-\eta^2)=\frac{\rho_1}{2}\frac{Q_1^2}{\eta+h_1}\left[1-\frac{1}{3}\left(\frac{d\eta}{dx}\right)^2\right]+\frac{\rho_2}{2}\frac{Q_2^2}{h_2-\eta}\left[1-\frac{1}{3}\left(\frac{d\eta}{dx}\right)^2\right]. \tag{17}$$

3. Solitary wave solution

Assuming waves of solitary type, we have at infinity a uniform flow with constant velocity c.

$$Q_1=ch_1\,,\quad Q_2=ch_2, \tag{18}$$

$$R_1=\frac{\rho_1}{2}c^2+\rho_2 gh_2\,, \tag{19}$$

$$R_2=\frac{\rho_2}{2}c^2+\rho_2 gh_2\,, \tag{20}$$

$$S=(\rho_1 h_1+\rho_2 h_2)c^2 + \frac{g}{2}(\rho_1 h_1{}^2+\rho_2 h_2{}^2)+\rho_2 g h_1 h_2. \tag{21}$$

From Eq.'s (17), (18), (19), (20) and (21), we obtain

$$\frac{c^2}{3}[(\rho_1 h_1{}^2-\rho_2 h_2{}^2)\eta-h_1h_2(\rho_1h_1+\rho_2h_2)]\left(\frac{d\eta}{dx}\right)^2 +(\rho_1-\rho_2)g\eta^4+(\rho_1-\rho_2)[g(h_1-h_2)-c^2]\eta^3 +[(\rho_1h_2+\rho_2h_1)c^2-(\rho_1-\rho_2)gh_1h_2]\eta^2=0,$$

which can be written in a simpler form by putting

$$\eta=h_1\zeta\,,\quad x=h_1\xi\,,\quad A=h_1\alpha,$$

$$h_2/h_1=r\,,\quad \rho_2/\rho_1=s,$$

$$c_0{}^2=\frac{g(\rho_1-\rho_2)h_1h_2}{\rho_1h_2+\rho_2h_1},\quad F^2=c^2/c_0{}^2\,.$$

That is,

$$\frac{M}{2}\left(\frac{d\zeta}{d\xi}\right)^2+K=0, \tag{22}$$

where

$$M=\frac{2F^2r^2(1+rs)}{3(r+s)},$$

$$K=\frac{r(1-F^2)\zeta^2+\left[r-1+\dfrac{r(1-s)}{r+s}F^2\right]\zeta^3-\zeta^4}{1+\dfrac{r^2s-1}{r(1+rs)}\zeta}.$$

Eq. (22) can be interpreted as the motion of a particle with mass M and zero total energy in a field whose potential is given by K. The curve for K is shown in Fig. 2. It is seen that solitary waves are possible when K is negative through $\eta=0$ and $\eta=\alpha$. For this, the leading term of K must be negative:

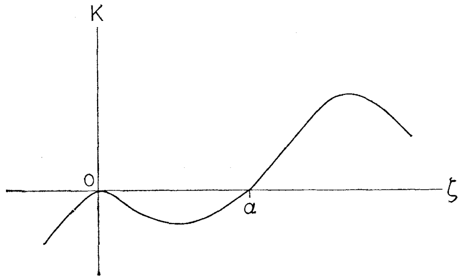

Fig. 2. The potential K as a function of ζ. (The scales are arbitrary.)

$$F^2>1\,.$$

Since α represents the amplitude of a solitary wave, $K=0$ at $\zeta=\alpha$ should provide the dispersion relation (Note that α is also related to horizontal scale as will be shown in Fig. 4):

$$r(1-F^2)+\left[r-1+\frac{r(1-s)}{r+s}F^2\right]\alpha-\alpha^2=0,$$

or

$$F^2=\frac{(1+\alpha)\left(1-\dfrac{\alpha}{r}\right)}{1-\dfrac{1-s}{r+s}\alpha}\,. \tag{23}$$

From Eq. (22) we can also determine the upper limit for the wave height:

$$\alpha\leqq\frac{1}{2}\left[r-1+\frac{r(1-s)}{r+s}F^2\right]. \tag{24}$$

Note that if the difference of the densities of the two layers is small ($s\sim1$),

$$\alpha\leqq\frac{1}{2}(r-1)\,.$$

Figure 3 shows the dispersion relation for various parameters. For comparison, the dispersion curves of the KdV and BENJAMIN-ONO solitary waves are also shown.

The shape of the wave is known from Eq. (22) which can be rewritten as:

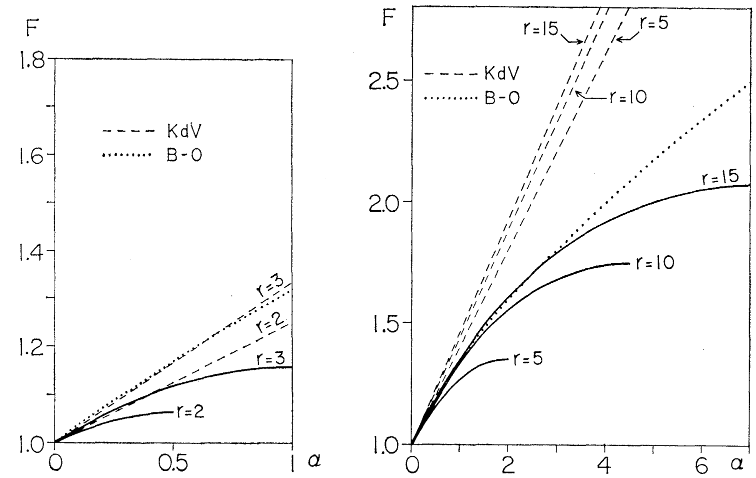

Fig. 3. The dispersion curves for several parameters of r and α, with $s=0.98$. The corresponding curves for the KdV (broken line) and BENJAMIN-ONO (dotted line) theories are based on Eq.'s (A2) and (A4).

$$\sqrt{\frac{2}{M}}d\xi=\int\frac{\sqrt{1+B\zeta}}{\zeta\sqrt{C+D\zeta-\zeta^2}}d\zeta$$
$$=B\int\frac{d\zeta}{\sqrt{(1+B\zeta)(C+D\zeta-\zeta^2)}}$$
$$+\int\frac{d\zeta}{\zeta\sqrt{(1+B\zeta)(C+D\zeta-\zeta^2)}}=E_1+E_3\,,$$

where

$$B=\frac{r^2s-1}{r(1+rs)},\quad C=r(1-F^2)\,,$$
$$D=r-1+\frac{r(1-s)}{r+s}F^2,$$

and E_1 and E_3 are elliptic integrals of the first and third kind (See, e.g. ABRAMOWITZ and STEGUN, 1965). Some examples of the wave form are shown in Fig. 4.

4. Discussions and concluding remarks

In the two-layer model of real ocean, s is usually close to unity and in that case, both the wave form and the dispersion relation depend little on s. Thus all the calculations for Figs. 3 and 4 were done for a fixed value of $s=0.98$. It is to be noted that the variation of F^2 is nearly quadratic in α (Ev. 23). Fig. 3 shows that the dispersion curve of the obtained solitary wave is substantially different from either that of the KdV or BENJAMIN-ONO soliton except for small α. It should be noted that the upper limit exists for α due to the inequality (24), but this is a necessary condition for the greatest wave height and whether or not it is also sufficient remains to be studied.

The wave profile of the present result is also quite different from the existing two theories (Fig. 4). However, the discrepancy between the obtained and the KdV solitary wave forms becomes smaller for the smaller amplitude. This can be also known from Eq. (22). That is, if $|\zeta|\ll 1$ is assumed, K in Eq. (22) is approximated by a cubic equation of ζ, which is an integral form of the KdV equation (BENJAMIN and LIGHTHILL 1954, FENTON 1972). In this sense the obtained wave is a generalized form of the KdV soliton for large amplitude.

For Q_1, Q_2, R_1, R_2 and S, it is possible to choose values other than those given by (18), (19), (20), and (21). Then, in general, a linear term and a constant will be added to the numerator of K, so that the solution of Eq. (22) will be a periodic wave of large amplitude which is a generalization of the cnoidal wave (see e.g. BENJAMIN and LIGHTHILL 1954).

The two-layer model presented here may be too much simplified to be directly applied to the real ocean where stratification plays an important role. However, some laboratory experiments suggest that such large amplitude waves can be

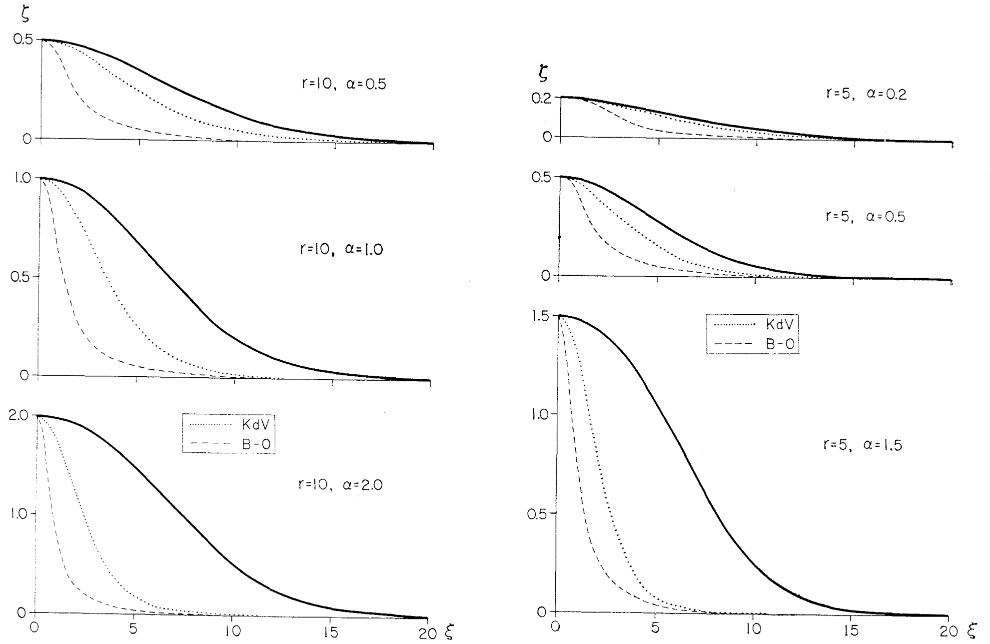

Fig. 4. Profiles of the solitary wave for various parameters of α and r, with $s=0.98$. The KdV (dotted line) and BENJAMIM-ONO (broken line) soliton forms calculated by Eq,'s (A1) and (A3) are also plotted.

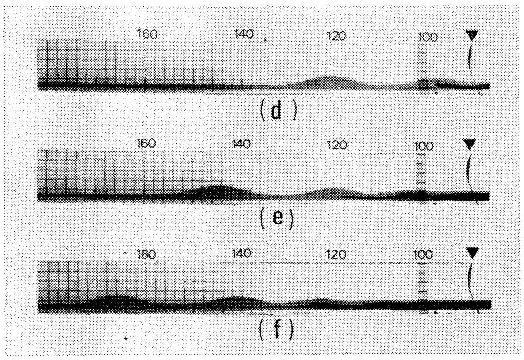

Fig. 5. Possible formation of large amplitude waves. (d), (e) and (f) shows a time sequence of the waves advancing to the left (After MANABE, 1984).

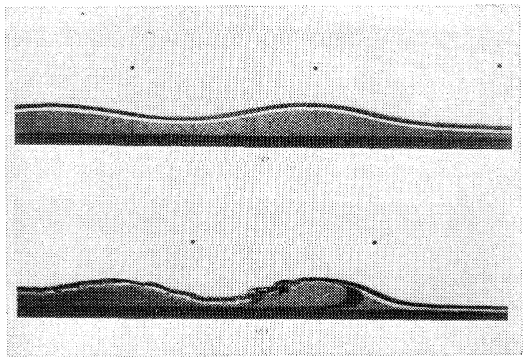

Fig. 6. Possible formation of large amplitude waves. The waves in (a) and (b) are produced in different depth ratios and they are advancing to the right (After WOOD and SIMPSON, 1984).

in fact generated. For instance, MAXWORTHY (1980) reported that large amplitude solitary waves were formed after gravitational collapse and mixing of stratified fluids. Manabe's (1984) experiment, although her main interest was on the behaviour of intruding density front using two miscible fluids, clearly shows a series of large bumps each of which resembles in shape the solitary wave discussed above (Fig. 5). Similar feature is also seen in WOOD and Simpson's (1984) experiment (Fig. 6). The front half of a bump from Manabe's experiment is drawn in Fig. 7 to be compared with theory. Agreement is fairly good.

In summary, a non-linear integral equation has been derived to describe a steady motion in a two-layer fluid system. This equation was expanded to an approximation to provide the

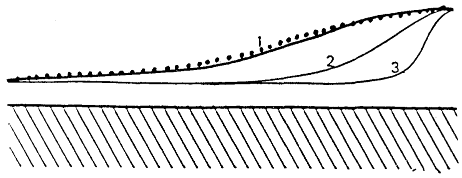

Fig. 7. Comparison of the theory (dotted line) with Manabe's experimental data (solid line 1). Corresponding KdV (solid line 2) and BENJAMIN-ONO (solid line 3) theoretical curves are also shown.

analytic solution of a solitary wave of large amplitude, which is different from either KdV or BENJAMIN-ONO soliton. The dispersion relation, with a necessary condition for the greatest wave amplitude, was also obtained.

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Appendix

The KdV equation for the two layer fluid system has the solitary wave solution of the form

$$=A\mathrm{sech}^2\frac{x-ct}{\lambda}, \qquad \text{(A1)}$$

with

$$c=c_0+\frac{Ac_1}{3},\ A\lambda^2=12\frac{c_2}{c_1}, \qquad \text{(A2)}$$

where

$$c_0{}^2=\frac{g(\rho_2-\rho_2)h_1h_2}{\rho_1h_2+\rho_2h_1},$$

$$c_1=\frac{3c_0(r^2-s)}{2h_1r(r+s)},$$

$$c_2=\frac{c_0h_1{}^2r(rs+1)}{6(r+s)},$$

$$s=\frac{\rho_2}{\rho_1} \text{ and } r=\frac{h_2}{h_1}.$$

The BENJAMIN-ONO equation gives the solitary wave profile of the Lorentzian type for internal wave motion of infinite extent. ($h_2\to\infty$, $h_1=h$)

$$\eta=\frac{A\lambda^2}{(x-ct)^2+\lambda^2}, \qquad \text{(A3)}$$

with

$$c=c_0\sqrt{1+\frac{3}{4}\frac{A}{h}},\ \lambda=\frac{4}{3}\frac{h^2}{rA}, \qquad \text{(A4)}$$

where

$$c_0{}^2=\frac{(\rho_1-\rho_2)gh}{\rho_1}.$$