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Theoretical elucidation of generation of a soliton on the interface of two-layer fluid system with equal depth with slightly different densities*

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Abstract: Analytical treatment is conducted to seek the soliton solution along an interface of two fluid system with equal depth and slightly different densities in the sea. There results a solitary wave of very small amplitude and with very long horizontal scale. If the density difference is somewhat large, the amplitude increases. In any case, the upper limit of amplitude exists. The case when the depth of upper layer is slightly larger than that of lower layer is also considered.

1. Introduction

Interfacial solitary wave which gives rise to on an interface of two-layer fluid system of equal depth of which densities are $\rho+\Delta\rho$ in the lower and ρ in the upper layer has not been solved theoretically yet (ROBERTS, 1975). Various experiments conducted by MIYATA (personal contact) pertaining to this phenomenon revealed generation of soliton when the upper layer is thicker than the lower, and revealed no conspicuous soliton when the thickness of both layer is equal.

MIYATA (1985) also studied the interfacial soliton theoretically by solving the nonlinear differential equation numerically, using the elliptic integral and obtained elevated soliton of large amplitude when the upper layer is deeper than lower layer and the density difference is small. The author tried here to seek the exact solutions of interfacial soliton in the case of equal depth, using the mathematical method applied by MIYATA by means of improved and completely analytical procedure.

In section 2, the fundamental dynamical equation is derived by considering horizontal constancy of a flow force in moving fluid using the complex potential method (LAMB, 1932). The resulting non-dimensional equation is the so-called nonlinear ordinary differential equation of polynomial class which is analytically integrable in special case (INCE, 1956).

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In section 3, the solution of the above equation is expressed in terms of integral of irrational function when depth ratio of two layers r is equal to 1, and $\sigma=\Delta\rho/\rho$ is arbitrary. In this case, elevation of the interface is assumed to be smaller than 1. We introduce the square of internal Froude number $F_i^2=c^2/gh\sigma$, where h is the depth of each layer, g acceleration of gravity and $\sigma=\Delta\rho/\rho$. F_i^2 is a function of wave height A, hence if F_i^2 increases with A $(0<A<1)$, steady soliton solution exists, but if it decreases with increasing A, physically such a soliton is unrealistic. We can prove mathematically that there exists always narrow domain of F_i^2 or A with $dF_i^2/dA>0$. In other words, this means that there is a limit of amplitude of a soliton for prescribed value of σ. Soliton profile is slightly elevated and very long horizontally.

When A is small, neglecting the fourth power of A and so on, the solution is given by elliptic functions. Numerical example is given for $\sigma=0.02$, being relevant value for research of two layers system in the sea. For larger σ, the amplitude tends to increase.

If the ratio of depth of the upper layer to the lower layer is slightly larger than 1, an elevated soliton with small amplitude also exists. As an example, we give this ratio to be $1/\sqrt{1-\sigma}$ which simplifies the fundamental equation and the solution can be easily expressed by elementary integral.

2. Construction of the fundamental equations

So far, we consider the two layers system

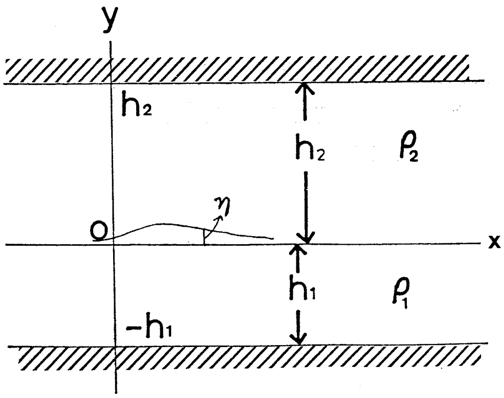

Fig. 1. Cartesian coordinate system, x-axis taking along the interface and y-axis taking vertically upwards from the origin which locates on the interface.

with densities ρ_1 and ρ_2 ($\rho_1 > \rho_2$) and the constant depth of h_1 and h_2 ($h_2/h_1 = r$) for lower and upper layer, respectively (Fig. 1). We assume the fluid extends horizontally to infinity, and the upper and bottom boundaries are bounded by rigid plane surface. Cartesian coordinates are taken, x-axis being horizontal and y-axis vertical and the origin locating on the interface.

Now let introduce total flow force S to which the vertical surface with unit width in the fluid is subject,

$$S=\int_{-h_1}^{\eta}(p_1+\rho_1 u^2)dy + \int_{\eta}^{h_2}(p_2+\rho_2 u_2^2)dy, \qquad (2.1)$$

where η is the elevation of the interface, ρ_n ($n=1,2$) densities of fluid, p_n and u_n are dynamical pressure and horizontal fluid velocity due to solitary wave motion, respectively. The Bernoulli's equations are

$$p_n+\rho_n gy+\frac{\rho_n}{2}(u_n^2+v_n^2)=K_n, \quad (n=1,2), \qquad (2.2)$$

where v_n's are vertical velocity and K_n's are constant. Upon substituting p_n of (2.2) into (2.1), we obtain

$$\begin{aligned}S=&K_1(\eta+h_1)-K_2(\eta-h_2)\\&-\frac{\rho_1 g}{2}(\eta^2-h_1^2)-\frac{\rho_2 g}{2}(h_2^2-\eta^2)\\&+\frac{\rho_1}{2}\int_{-h_1}^{\eta}(u_1^2-v_1^2)dy\\&+\frac{\rho_2}{2}\int_{\eta}^{h_2}(u_2^2-v_2^2)dy,\end{aligned} \qquad (2.3)$$

which is also given by MIYATA (1985). Since the flow force S is conserved horizontally then, the fundamental conception to derive the equation of motion is $S=S_\infty$, where S_∞ is the flow force at $x=\pm\infty$.

Assuming a stationary inviscid fluid motion, we introduce the complex velocity potential $\chi_n=\phi_n+i\psi_n (n=1,2)$ and complex coordinate $z=x+iy$, hence we obtain

$$\frac{d\chi_n}{dz}=u_n-iv_n, \quad (n=1,2). \qquad (2.4)$$

Near the bottom we expand $u_n(x,y)$ in Taylor series of $y-h_1$ and h_2-y for $n=1$ and $n=2$, respectively, and using

$$u_n(x,y)=\frac{\partial\phi_n}{\partial x}=Re\frac{d\chi_n}{dz},$$

we obtain (LAMB, 1932)

$$\left.\begin{aligned}u_1(x,y)=&u_1(x,\ -h_1)\\&-\frac{1}{2!}(y+h_1)^2\frac{d^2}{dx^2}u_1(x,\ -h_1)\\&+\frac{1}{4!}(y+h_1)^4\frac{d^4}{dx^4}u_1(x,-h_1)+\dots,\\u_2(x,y)=&u_2(x,h_2)\\&-\frac{1}{2!}(h_2-y)^2\frac{d^2}{dx^2}u_2(x,h_2)\\&+\frac{1}{4!}(h_2-y)^4\frac{d^4}{dx^4}u_2(x,h_2)+\dots.\end{aligned}\right\} \qquad (2.5)$$

Since we consider a solitary wave motion *a priori*, vertical motion is very small compared with horizontal, the fourth order terms in the above equations can be neglected. Now we can express $u_1(x,-h_1)$ and $u_2(x,h_2)$ in terms of horizontal flow rates

$$Q_1=\int_{-h_1}^{\eta}u_1 dy=\int_{-h_1}^{\eta}\left[1-\frac{(y+h_1)^2}{2!}\frac{d^2}{dx^2}+O\left(\frac{\eta+h_1}{L}\right)^4\right]u_1(x,-h_1)dy,$$

L being the characteristic horizontal length, integrating this with respect to y we obtain

$$Q_1=(\eta+h_1)\left[1-\frac{(\eta+h_1)^2}{3!}\frac{d^2}{dx^2}+\dots\right]+u_1(x,-h_1), \qquad (2.6)$$

and similarly,

$$Q_2=\int_\eta^{h_2}\left[1-\frac{(h_2-y)^2}{2!}\frac{d^2}{dx^2}+\dots\right]u_2(x,h_2)dy$$
$$=(h_2-\eta)\left[1-\frac{(h_2-\eta)^2}{3!}\frac{d^2}{dx^2}+\dots\right]u_2(x,h_2). \tag{2.7}$$

Solving u_1 and u_2 from (2.6) and (2.7) and considering Q_1 and Q_2 are conserved horizontally, we obtain

$$\left.\begin{aligned}
&u_1(x,-h_1)\\
&\quad=\left[1+\frac{(\eta+h_1)^2}{3!}\frac{d^2}{dx^2}\right]\frac{Q_1}{\eta+h_1}\\
&\quad=\frac{Q_1}{h_1+\eta}\left[1+\frac{1}{3}\left(\frac{d\eta}{dx}\right)^2\right.\\
&\quad\left.-(h_1+\eta)\frac{d^2\eta}{dx^2}\right],\\
&u_2(x,h_2)=\frac{Q_2}{h_2-\eta}\left[1+\frac{1}{3}\left(\frac{d\eta}{dx}\right)^2\right.\\
&\quad\left.-(h_2-\eta)\frac{d^2\eta}{dx^2}\right],\\
&v_1^2=\frac{Q_1}{(\eta+h_1)}\left(\frac{d\eta}{dx}\right)^2,\\
&v_2^2=\frac{Q_2}{(h_2-\eta)^2}\left(\frac{d\eta}{dx}\right)^2.
\end{aligned}\right\} \tag{2.8}$$

At infinity $x=\pm\infty$, we assume $u_1=u_2=c$, $p_1=p_2=0$, $v_1=v_2=0$ and $\eta=0$, then we have

$$K_1=\rho_2gh_2+\frac{\rho_1}{2}c^2,$$
$$K_2=\rho_2gh_2+\frac{\rho_2}{2}c^2,$$
$$Q_n=ch_n,\ (n=1,2). \tag{2.9}$$

Substituting K_1 into (2.2) for $n=1$, we have

$$\rho_2gh_2+\frac{\rho_1}{2}c^2=p_1+\rho g\eta+\frac{\rho_1}{2}(u_1^2+v_1^2)$$

which reduces to $p_1=\rho_2gh_2-\rho_1g\eta$ for $x\to\pm\infty$. Hence, the flow force $S_{1\infty}$ is given by

$$\left.\begin{aligned}
&S_{1\infty}=\int_{-h_1}^{0}(p_1+\rho_1c^2)dy=\rho_1c^2h_1\\
&\quad+\rho_2gh_1h_2+\frac{1}{2}\rho_1gh_1^2,\\
&\text{and similarly}\\
&S_{2\infty}=\int_0^{h_2}(p_2+\rho_2c^2)dy=\int_0^{h_2}(\rho_2c^2+\rho_2gh_2\\
&\quad-\rho_2gy)dy=\rho_2h_2c^2+\frac{\rho_2g}{2}h_2^2.
\end{aligned}\right\} \tag{2.10}$$

Upon substituting (2.8) into (2.3) and using $S=S_\infty=S_{1\infty}+S_{2\infty}$, we obtain the dynamical equation in non-dimensional form as

$$N(1+\zeta)(r-\zeta)\frac{d^2\zeta}{d\xi^2}+M(1-B\zeta)\left(\frac{d\zeta}{d\xi}\right)^2$$
$$=\zeta^4-D\zeta^3-C\zeta^2, \tag{2.11}$$

where $x=h_1\xi$, $\zeta=h_1\eta$ and

$$\left.\begin{aligned}
&N=4F_i^2(1-r^2+\sigma r^2),\\
&B=\frac{1-r^2+\sigma r^2}{r(1+r-\sigma r)},\\
&D=r-1+\sigma F_i^2,\\
&M=\frac{F_i^2}{3}r(1+r-\sigma r),\\
&C=r-F_i^2(1+r-\sigma),
\end{aligned}\right\} \tag{2.12}$$

MIYATA (1985) also obtained the equation like (2.11), however the term $d^2\zeta/d\xi^2$ is in defect. This term is not small compared to $(d\zeta/d\xi)^2$ and so cannot be ignored.

3. Integration of fundamental equation (2.11)

To solve (2.11), we use the conventional method, that is to say the so-called variation of integral constant. First we integrate the homogeneous equation of (2.11), namely the equation without the righthand side. The integral constant K including in this solution is then considered to be the function of ζ, that is to say

$$\frac{N}{2}(1+\zeta)(r-\zeta)\frac{dK}{d\zeta}$$
$$=(\zeta^4-D\zeta^3-C\zeta^2)\exp\left[\frac{M}{N}f(\zeta)\right], \tag{3.1}$$

where we get the following

$$f(\zeta)=\log\frac{(1+\zeta)^{\alpha_1}}{(r-\zeta)^{\alpha_2}},$$

and so

$$\exp\left[\frac{M}{N}f(\zeta)\right]=\left[\frac{(1+\zeta)^{\alpha_1}}{(r-\zeta)^{\alpha_2}}\right]^{\frac{M}{N}}=\frac{(1+\zeta)^{\alpha_1}}{(r-\zeta)^{\alpha_2}}.$$

Integrating (3.1), we obtain

$$K(\zeta)=\frac{2}{N}\int\frac{(1+\zeta)^{\alpha_1}}{(r-\zeta)^{\alpha_2}}\frac{\zeta^4-D\zeta^3-C\zeta^2}{(1+\zeta)(r-\zeta)}d\zeta, \tag{3.2}$$

where $K(\zeta)$ must be positive (see below).

The solution to the above-mentioned homogeneous equation is easily obtained as follows,

$$\left(\frac{d\zeta}{d\xi}\right)^2=K(\zeta)\exp\left[-\frac{M}{N}f(\zeta)\right], \qquad (3.3)$$

where

$$f(\zeta)=\int\frac{1-B\zeta}{(1+\zeta)(r-\zeta)}d\zeta=\log\frac{(1+\zeta)^{a_1}}{(r-\zeta)^{a_2}}.$$

Finally, inserting (3.2) into (3.3) and taking square root, we have

$$\xi=\pm\sqrt{\frac{N}{2}}\int\frac{(1+\zeta)^{\alpha_1}d\zeta}{(r-\zeta)^{\alpha_2}\sqrt{J(\zeta)}}, \qquad (3.4)$$

where

$$J(\zeta)=\int\frac{(1+\zeta)^{\alpha_1-1}}{(r-\zeta)^{\alpha_2+1}}(\zeta^4-D\zeta^3-C\zeta^2)d\zeta. \qquad (3.5)$$

Powers a_1 and a_2 included in $f(\zeta)$ are

$$a_1=\frac{1+B}{1-r} \text{ and } a_2=\frac{1-rB}{1+r},$$

respectively, then we obtain

$$\left.\begin{aligned}\alpha_1&=\frac{a_1M}{N}=\frac{1}{12(1-r^2+\sigma r^2)}\\ \alpha_2&=\frac{a_2M}{N}=\frac{r^2(1-\sigma)}{12(1-r^2+\sigma r^2)}\\ &=r^2(1-\sigma)\alpha_1.\end{aligned}\right\} \qquad (3.6)$$

In general α_1 and α_2 are not integer, hence the integrands of (3.4) and (3.5) are irrational and we cannot integrate rigorously by elementary functions.

When $r=1$, the formulae (2.12) reduce to

$$\left.\begin{aligned}&N=4\sigma F_i^2,\quad M=\frac{2-\sigma}{3}F_i^2,\\ &B=\frac{\sigma}{2-\sigma},\quad C=1-(2-\sigma)F_i^2,\\ &D=\sigma F_i^2,\end{aligned}\right\} \qquad (3.7)$$

and $\alpha_1=1/12\sigma$, $\alpha_2=(1-\sigma)/12\sigma$. Now let consider the evaluation of (3.5). Assuming $\zeta<1$, expanding the integrand of (3.5) in power series of ζ we get

$$\zeta^2(C+D\zeta-\zeta^2)(1+\beta_1\zeta+\beta_2\zeta^2+\beta_3\zeta^3+\cdots), \qquad (3.8)$$

where

$$\begin{aligned}\beta_1&=\frac{2-\sigma}{12\sigma},\\ \beta_2&=\frac{4-4\sigma+277\sigma^2}{288\sigma^2}\\ \beta_3&=\frac{(2-\sigma)(4-4\sigma+1117\sigma^2)}{10368\sigma^3}.\end{aligned} \qquad (3.9)$$

Replacing the integrand of (3.5) by (3.8) and integrating term by term, we obtain

$$\left.\begin{aligned}&J(\zeta)=\zeta^3\bar{\varphi}(\zeta),\\ &\bar{\varphi}(\zeta)=-(d_0+d_1\zeta+d_2\zeta^2+d_3\zeta^3+\cdots),\end{aligned}\right\} \qquad (3.10)$$

where

$$d_0=\frac{C}{3}=d_{01}-d_{02}F_i^2;\quad d_{01}=\frac{1}{3},\quad d_{02}=\frac{2-\sigma}{3}, \qquad (3.11)$$

$$d_1=\frac{\beta_1C+D}{4}=d_{11}-d_{12}F_i^2;\quad d_{11}=\frac{\beta_1}{4}=\frac{2-\sigma}{48\sigma},\quad d_{12}=\frac{1}{4}\{(2-\sigma)\beta_1-\sigma\}=\frac{4-4\sigma-11\sigma^2}{48\sigma}, \qquad (3.12)$$

$$\left.\begin{aligned}d_2&=\frac{\beta_2C+\beta_1D-1}{5}=d_{21}-d_{22}F_i^2;\\ d_{21}&=\frac{1}{5}(\beta_2-1)=\frac{4-4\sigma-11\sigma^2}{1440\sigma^2},\\ d_{22}&=\frac{1}{5}\{\beta_2(2-\sigma)-\beta_1\sigma\}\\ &=\frac{(2-\sigma)(4-4\sigma+203\sigma^2)}{1440\sigma^2}\end{aligned}\right\} \qquad (3.13)$$

$$\left.\begin{aligned}d_3&=\frac{1}{6}(\beta_3C+\beta_2D-\beta_1)=d_{31}-d_{32}F_i^2:\\ d_{31}&=\frac{1}{6}(\beta_3-\beta_1)\\ &=\frac{(2-\sigma)(4-4\sigma+253\sigma^2)}{62208\sigma^3},\\ d_{32}&=\frac{1}{6}\{(2-\sigma)\beta_3-\beta_2\sigma\}\\ &=[(2-\sigma)^2(4-4\sigma+1117\sigma^2)-36\sigma^2\\ &\quad\times(4-4\sigma+277\sigma^2)]\div62208\sigma^3\end{aligned}\right\} \qquad (3.14)$$

According to (3.4), since N is positive $J(\zeta)$ must be also positive so as ξ to be real, for this d_0 must be negative or $C<0$, that is to say it is necessary that $F_i^2>1/(2-\sigma)$. This fact implies that an elevated soliton ($1>A>0$) exists physically. Moreover, we must cheque whether $F_i^2=c^2/gh\sigma$ increases with A or not, in other words, it is necessary $dF_i/dA>0$ in order to real soliton exist, because the higher is the soliton, the larger the phase velocity c must be.

At the top of profile $\zeta(\xi)$ of a soliton, the tangent must be horizontal or $d\zeta/d\xi=0$. To satisfy this condition it is clear $d\xi/d\zeta=\infty$ for $\zeta=A$ or $J(A)=A\bar{\varphi}(A)=0$ or $\bar{\varphi}(A)=0$. We can solve F_i^2 from this equation as a function of A, and obtain the dispersion relation as

$$F_1^2=\frac{d_{01}+d_{11}A+d_{21}A^2+d_{31}A^3+\cdots}{d_{02}+d_{12}A+d_{22}A^2+d_{32}A^3+\cdots}. \quad (3.15)$$

Differentiating F_i^2 with respect to A, we have

$$\lim_{A\to 0}\frac{dF_i^2}{dA}=d_{11}d_{02}-d_{12}d_{01}.$$

In order to make dF_i^2/dA positive we must have

$$d_{11}d_{02}-d_{12}d_{01}>0. \quad (3.16)$$

Making use of (3.11) and (3.12), lefthand side of the above expression becomes

$$\frac{(2-\sigma)^2}{144\sigma}-\frac{4-4\sigma-12\sigma^2}{144\sigma}=\frac{\sigma}{12}>0,$$

hence, (3.16) is always satisfied. This fact provides the existence of a real soliton in the neighborhood of $A=0$. F_i^2 has a maximum value for $A=A_m$ and for A larger than A_m no soliton exists physically. ROBERTS (1975) has described in her book that no theory has been discovered pertaining to the present problem. However, we believe that the present discussion resolves the pending problem.

4. Integration of (3.4) when $\sigma=0.02$

$\sigma=0.02$ is relevant value for dynamical problem in the sea. Now the dispersion relation (3.15) is illustrated in Fig. 2. For $A>0.02$, the curve is invalid, since F_i^2 decreases with increasing A, hence solitons occur in the very narrow interval of F_i^2, that is to say $0.50505<F_i^2<0.505096$. Since A is very small $J(\zeta)$ of (3.10) is approximately written by

$$J(\zeta)=\zeta^3\bar{\varphi}(\zeta)\equiv\zeta^3(|d_0|-d_1\zeta-d_2\zeta^2-d_3\zeta^3). \quad (4.1)$$

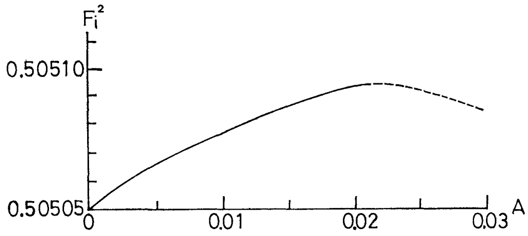

Fig. 2. Dispersion curve when $r=1$, $\sigma=0.02$. The maximum value of A is about 0.02, and of F_i^2 is 0.505096, the part of thickline of the curve being available.

Computing for $F_i^2=0.50508$ the numerical values of d_n's by (3.11)~(3.14) and solving $\bar{\varphi}(\zeta)=0$, we obtain $\zeta_1=0.00901$, $\zeta_2=0.03162$ and $\zeta_3=-0.05196$, among those roots, the amplitude of soliton corresponds to ζ_1.

Now we expand the following factor in the integrand (3.4) as

$$\frac{(1+\zeta)^{\alpha_1}}{(1-\zeta)^{\alpha_2}}=1+r_1\zeta+r_2\zeta^2+r_3\zeta^3+O(\zeta^4), \quad (4.2)$$

where

$$\left.\begin{aligned} r_1&=\frac{2-\sigma}{12\sigma},\quad r_2=\frac{4-4\sigma-11\sigma^2}{2888\sigma^2},\\ r_3&=\frac{(2-\sigma)(4-4\sigma-253\sigma^2)}{10368\sigma^3}.\end{aligned}\right\} \quad (4.3)$$

Thus from (3.4) we obtain

$$\xi=\pm\sqrt{2\sigma F_i^2}\times(I_{-1}+r_1I_0+r_2I_1+r_3I_2+\cdots), \quad (4.4)$$

where

$$I_n=\int_A^{\zeta}\frac{\zeta^n}{\sqrt{\varphi(\zeta)}}d\zeta,\quad \varphi(\zeta)=\zeta\bar{\varphi}(\zeta). \quad (4.5)$$

The integral variable is now transformed to w by

$$\zeta=\frac{|d_0|}{4w+d_1/3}, \quad (4.6)$$

hence we obtain

$$\left.\begin{aligned} \zeta\bar{\varphi}(\zeta)=\varphi(\zeta)&=\frac{16|d_0|^2}{(4w+d_1/3)^4}\\ &\times 4(w-e_1)(w-e_2)(w-e_3),\\ \text{where}&\\ e_r=|d_0|/4\zeta_r&-d_1/12,\quad r=1,2,3,\end{aligned}\right\} \quad (4.7)$$

then, numerical value of e_r's are $e_1=3.4065\times10^{-4}$, $e_2=-4.625\times10^{-4}$ and $e_3=-2.9429\times10^{-4}$.

By differentiating $\zeta^n\sqrt{\varphi(\zeta)}$ with respect to ζ and integrating, we obtain the following recurrence formula for I_n,

$$\zeta^n\sqrt{\varphi(\zeta)}=\frac{2n+1}{2}|d_0|I_n-(n+1)d_1I_{n+1}-\frac{2n+3}{2}d_2I_{n+2}-(2n+4)d_3I_{n+3}. \quad (4.8)$$

Putting $n=-1$ we obtain

$$2d_3I_2=\frac{|d_0|}{2}I_{-1}-\frac{d_2}{2}I_1-\frac{\sqrt{\varphi(\zeta)}}{\zeta}. \quad (4.9)$$

Upon substituting (4.9) into (4.4), it follows

$$\xi(u)=\pm\sqrt{2\sigma F_i^2}\left[\left(1+\frac{r_3|d_0|}{4d_3}\right)I_{-1}+r_1I_0+\left(r_2-\frac{r_3d_2}{4d_3}\right)I_1-\frac{r_3}{2d_3}\times\left\{\frac{(|d_0|-d_1\zeta-d_2\zeta^2-d_3\zeta^3)}{\zeta}\right\}^{1/2}\right]. \quad (4.10)$$

We introduce the second transformation

$$w=e_1+\frac{(e_1-e_2)\mathrm{sn}^2u}{\mathrm{cn}^2u},\ (\mathrm{mod.}\ k) \quad (4.11)$$

modulus of sn$u$ being defined by

$$k^2=\frac{e_2-e_3}{e_1-e_3},\ k'^2=1-k^2. \quad (4.12)$$

After some analytic manipulations we can reduce the following:

$$\left.\begin{aligned} I_0&=-u/\sqrt{e_1-e_3},\\ I_{-1}&=-\frac{12e_1+d_1}{3|d_0|\sqrt{e_1-e_3}}u-\frac{4\sqrt{e_1-e_3}}{|d_0|}\left\{\frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}u}-E(u)\right\},\\ I_1&=\frac{-3|d_0|}{\sqrt{e_1-e_3}\,(12e_1+d_1)}\times\left\{u-\frac{\mathrm{dn}a}{k^2\,\mathrm{sn}a\,\mathrm{cn}a}\Pi(u,a)\right\}.\end{aligned}\right\} \quad (4.13)$$

where $\Pi\ (u,a)$ is defined by

$$\Pi(u,a)=\int_0^u\frac{k^2\,\mathrm{sn}a\,\mathrm{cn}a\,\mathrm{dn}a\,\mathrm{sn}^2u\,du}{1-k^2\,\mathrm{sn}^2a\,\mathrm{sn}^2u}=-\sum_{n=1}^{\infty}\frac{\sin(n\pi a/K)\sin(n\pi u/K)}{n\sinh(n\pi K'/K)}+\frac{\pi u}{K}\sum_{n=1}^{\infty}\frac{\sin(n\pi a/K)}{\sinh(n\pi K'/K)} \quad (4.14)$$

(WHITTAKER and WATSON, 1927; ABRAMOWITZ and STEGUN 1965), where a is determined by

$$\mathrm{sn}^2a=\frac{12e_2+d_1}{k^2(12e_1+d_1)}. \quad (4.15)$$

a is real and positive, since

$$1-\mathrm{sn}^2a=\frac{12(k^2e_1-e_2)-k'^2d_1}{k^2(12e_1+d_1)},$$

and we can show the numerator is positive, thus $\mathrm{sn}^2a<1$; actually, $a=1.7845$ for $\sigma=0.02$. K, K', and E are complete elliptic integrals of the first (K and K') and second (E) kind. Then $E(u)$ is represented by

$$E(u)=\frac{E}{K}u+\frac{\pi}{K}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi u}{K}}{\sinh\frac{2n\pi K'}{K}}. \quad (4.16)$$

Making use of (4.6) and (4.11), the elevation of the interface ζ is given by

$$\zeta(u)=\frac{3|d_0|\mathrm{cn}^2u}{12(e_1-e_2)+(12e_2+d_1)\mathrm{cn}^2u}. \quad (4.17)$$

(4.10) and (4.17) give profile of elevated soliton $\zeta(\xi)$ in terms of parameter u, which is illustrated in Fig. 3 for $\sigma=0.02$ and $F_i^2=0.50508$. For $u=0$, $\xi(0)=0$ and $\zeta(0)=3|d_0|/(12e_1+d_1)=A$, and for $u=K$, $\xi(K)=\infty$ and $\zeta(K)=0$. By computing the coefficients of three formulae of (4.13), we can see $I_1\ll I_{-1}$, since $|d_0|$ (say $|d_0|=1.95\times10^{-5}$ for $\sigma=0.02$ and $F_i^2=0.50508$) is generally very small, and the last term of (4.10) is also small compared with terms I_0 and I_{-1}, hence taking the negative sign of (4.10), $\xi(u)$ is expressed by approximate formula as follows:

$$\xi(u)\fallingdotseq-\sqrt{2\sigma F_i^2}(I_{-1}+r_1I_0), \quad (4.18)$$

or numerical form as

$$\xi(u)\fallingdotseq62.30u+734.55\times\left\{\frac{\mathrm{sn}u\,\mathrm{dn}u}{\mathrm{cn}u}-E(u)\right\}, \quad (4.19)$$

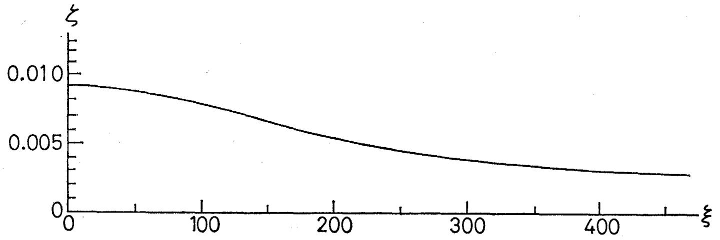

Fig. 3. Right half of the soliton profile when $\sigma=0.02$ and $F_i^2=0.50508$. Maximum height is 0.00901.

for $F_i^2=0.50508$. $\xi(u)$ is very large compared with $\zeta(u)$, hence the soliton profile is very flat, so it may not be easy to make field observation or realize a tank experiment. In the case of larger σ, the maximum amplitude increases, for example when $\sigma=1/3$ A_m is about 0.1 for $F_i^2=0.601$.

5. The case of $r=1/\sqrt{1-\sigma}$

Inserting $r=1/\sqrt{1-\sigma}$ into (2.12), the fundamental equation (2.11) reduces to

$$\frac{M}{2}\left(\frac{d\zeta}{d\xi}\right)^2=\zeta^2(\zeta^2-D\zeta-C),\ C<0, \quad (5.1)$$

or

$$\xi=\pm\sqrt{\frac{M}{2}}\int_A^{\zeta}\frac{d\zeta}{\zeta\sqrt{\zeta^2-D\zeta+|C|}}, \quad (5.2)$$

where

$$\left.\begin{aligned} M&=\frac{F_i^2}{3}(\sqrt{1-\sigma}+1-\sigma),\\ C&=\frac{1}{\sqrt{1-\sigma}}-\left(1-\sigma+\frac{1}{\sqrt{1-\sigma}}\right)F_i^2,\\ D&=\frac{1}{\sqrt{1-\sigma}}-1+\sigma F_i^2.\end{aligned}\right\} \quad (5.3)$$

Now (5.2) is elementally integrable giving

$$\xi=\sqrt{\frac{M}{2|C|}}\times\left(\log\frac{\zeta+\sqrt{|C|}-\sqrt{\zeta^2-D\zeta+|C|}}{\zeta-\sqrt{|C|}-\sqrt{\zeta^2-D\zeta+|C|}}-\log\frac{A+\sqrt{|C|}}{A-\sqrt{|C|}}\right), \quad (5.4)$$

where

$$A^2-DA+|C|=0. \quad (5.5)$$

From (5.4), when $\zeta=0$, $\xi=\infty$ and when $\zeta=A$, $\xi=0$, then a soliton exists for certain interval of F_i^2. Using expressions C and D given by (5.3), (5.5) becomes

$$A^2-\left(\frac{1}{\sqrt{1-\sigma}}-1+\sigma F_i^2\right)A+\left(1-\sigma+\frac{1}{\sqrt{1-\sigma}}\right)F_i^2-\frac{1}{\sqrt{1-\sigma}}=0,$$

or solving F_i^2 we obtain the dispersion relation as

$$F_i^2=\frac{1+(1-\sqrt{1-\sigma})A-\sqrt{1-\sigma}A^2}{1+(1-\sigma)^{3/2}-\sigma\sqrt{1-\sigma}A}. \quad (5.6)$$

C is negative, so we have $F_i^2>1/\{1+(1-\sigma)^{3/2}\}=0.50757(\sigma=0.02)$. In order to seek maximum value of $dF_i^2/dA=0$ for A, that is to say, such A is a root of

$$(1-\sigma)(1-\sqrt{1-\sigma})A^2-2\sqrt{1-\sigma}(2-\sigma-\sqrt{1-\sigma})A+(2-\sigma)(1-\sqrt{1-\sigma})=0 \quad (5.7)$$

Now assuming $\sigma=0.02$ one root of (5.7) is $A_m=0.01015$, and from (5.6) $\max F_i^2=0.50763$. As the result elevated soliton with amplitudes less than 0.01015 occur for F_i^2 lying between the narrow interval $0.50757<F_i^2<0.50763$.

6. Conclusion

(1) An elevated soliton solution exists on the interface of two layers of equal thickness. When the density difference ratio $\sigma=\Delta\rho/\rho$ is 0.02, the height of soliton is less than 0.02 of the depth of single layer, and F_i^2, the square of internal Froud number for which the soliton exists, lies in very narrow interval $0.50505<F_i^2<0.50510$, $F_i^2=0.5$ being the value of infinitesimal interfacial waves generating on the same interface.

(2) If the depth of upper layer is $1/\sqrt{1-\sigma}$ times deeper than the lower layer, the exact

solution can be obtained by simple elementary analysis. In this case the elevated maximum height is 0.01015, less than the case of exactly equal depth, however F_i^2 is somewhat larger.

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等深およびほぼ等深の２層流体の界面孤立波の解析解存在の検討

冨　永　政　英

要旨: 密度を異にし，等深の２層をなす流体の界面に孤立波が生ずるのかどうかは理論的に不明とされていたし，実験室でも確認は困難であった。著者は非粘性，定常運動の仮定のもとに非線型常微分程式を誘導し，密度差の比$\sigma=\Delta\rho/\rho$が任意に与えられるとき，盛り上りの極めて小さい孤立波の解を証明した。その振幅は単層の厚さの100分１ていどに過ぎない。また上層の厚さが下層の$1/\sqrt{1-\sigma}>1$倍の場合，σが小さいとやはり振幅小なる孤立波が存在する。